Interplay in various settings between shift invariant spaces, wavelets, and sampling

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ABSTRACT

Shift invariant spaces are common in the study of analysis, appearing, for example, as cornerstones of the theories of wavelets and sampling. The interplay of these three notions is discussed at length over $\mathbb{R}$, with the one-dimensional study providing motivation for later discussions of $\mathbb{R}^n$, locally compact abelian groups, and some non-abelian groups. Two fundamental tools, the so-called "bracket" as well as the Zak transform(s), are described, and their deep connections to the aforementioned areas of study are made explicit.

Keywords: shift invariant spaces, wavelets, sampling theory, harmonic analysis

1. $\mathbb{R}$, THE ONE-DIMENSIONAL SETTING

We begin by describing facts about $L^2(\mathbb{R})$. It is well known that the properties of translation invariant (closed) subspaces of $L^2(\mathbb{R})$ play an important role in Mathematical Analysis involving $L^2(\mathbb{R})$ — see, for example the classical text by Helson.[1] These spaces are made up of those basic closed subspaces which are generated by systems of functions $\mathcal{B}_\phi = \{\phi(\cdot - k) : k \in \mathbb{Z}\}$, where ϕ is a nonzero element of $L^2(\mathbb{R})$. Let ϕ_k denote $\phi(\cdot - k)$ and also let $\langle\phi\rangle$ denote

$$\langle\phi\rangle := \overline{\operatorname{span}\{\phi_k : k \in \mathbb{Z}\}}, \tag{1}$$

which we refer to as the *principal shift invariant space generated by* ϕ; that is, $\langle\phi\rangle$ is the closure in $L^2(\mathbb{R})$ of all finite linear combinations $\sum_{\text{finite}} \alpha_k \phi_k$, where $\alpha_k \in \mathbb{C}$.

Before moving forward, we should remark that the Fourier transform we will be using is

$$(\mathfrak{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

Let us state some of the properties of these spaces obtained in an article by Hernández, Šikić, Weiss, and Wilson:[2] if $\phi, \psi \in L^2(\mathbb{R})$, let

$$[\phi,\psi](\xi) := \sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi + j)\overline{\widehat{\psi}(\xi + j)},$$

which is an almost everywhere defined, 1-periodic function in $L^1([0,1)) = L^1(\mathbb{T})$ (here $\mathbb{T}$ denotes the 1-torus, $\hat{\mathbb{R}}/\mathbb{Z}$). We call $[\phi,\psi]$ the *bracket of* ϕ *and* ψ in $L^2(\mathbb{R})$. The bracket is a "generalized" inner product and is most useful for our purposes. For example, $\langle\phi\rangle \perp \langle\psi\rangle$ if and only if $[\phi,\psi] \equiv 0$ almost everywhere. It can be shown[2] that if $p_\phi = [\phi,\phi]$, then the weighted space $\mathcal{M}_\phi := L^2(\mathbb{T}, p_\phi)$ is "naturally" isometric to the space $\langle\phi\rangle$. The "natural" isometry $J_\phi : \mathcal{M}_\phi \to \langle\phi\rangle$ is, for each $m \in \mathcal{M}_\phi$, given by

$$J_\phi(m) = (m\widehat{\phi})^\vee. \tag{2}$$

We show that the properties of the system $\mathcal{B}_\phi$ correspond to properties of the weight p_ϕ.

(i) *Linear independence.* If ϕ is not the zero element of $L^2(\mathbb{R})$, then $\mathcal{B}_\phi$ is a linearly independent set. This follows from the fact that $J_\phi(e^{-2\pi i k\xi}) = \phi(\cdot - k)$ combined with the fact that $\{e^{-2\pi i k\xi} : k \in \mathbb{Z}\}$ is a linearly independent system if $p_\phi > 0$ on a set of positive measure.

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(ii) *Orthonormality.* $\mathcal{B}_\phi$ is an orthonormal basis for $\langle\phi\rangle$ if and only if $p_\phi \equiv 1$ almost everywhere.[2]

(iii) *Existence of orthonormal basis.* There exists $\psi \in \langle\phi\rangle$ such that $\mathcal{B}_\psi$ is an orthonormal basis of $\langle\phi\rangle$ if and only if $p_\phi > 0$ almost everywhere: just consider

$$\widehat{\psi}(\xi) = \frac{\mu(\xi)}{\sqrt{p_\phi(\xi)}}\widehat{\phi}(\xi),$$

where μ is 1-periodic and unimodular.[2]

We feel that it is very useful to cite a 2011 article[3] that derives the Plancherel properties of the Fourier transform in a very simple and efficient way in terms of the two *Zak transforms*, Z and $\widetilde{Z}$, defined by

$$(Zf)(x,\xi) = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi ik\xi}$$

and

$$(\widetilde{Z}g)(x,\xi) = \sum_{\ell\in\mathbb{Z}} g(\ell+\xi)e^{2\pi i\ell\xi},$$

when $f, g \in L^2(\mathbb{R})$. Since $f \in L^2(\mathbb{R})$, we have $\sum_{k\in\mathbb{Z}} |f(x+k)|^2 < \infty$ almost everywhere, and thus $(Zf)(x,\xi)$ is clearly defined almost everywhere as the Fourier series with coefficients $\{f(x+k) : k \in \mathbb{Z}\}$; the obvious analogous argument is valid for $\widetilde{Z}g$. The image $\phi(x,\xi) = (Zf)(x,\xi)$ is a function of the two variables x and ξ, is 1-periodic in ξ, and satisfies

$$\phi(x+m,\xi) = e^{2\pi im\xi}\phi(x,\xi), \text{ for } m \in \mathbb{Z}. \tag{3}$$

Moreover,

$$\int_0^1\int_0^1 |\phi(x,\xi)|^2 dx d\xi = \|f\|_{L^2(\mathbb{R})}.$$

Let $\mathfrak{M}$ be the space of all ϕ defined for $(x,\xi) \in \mathbb{R}^2$ which are 1-periodic in ξ, satisfy Equation 3, and have norm $\|\phi\|_{\mathfrak{M}} = \left(\int_0^1\int_0^1 |\phi(x,\xi)|^2 dx d\xi\right)^{1/2} < \infty$. A simple argument shows that Z is an isometry onto $\mathfrak{M}$.

Let $\widetilde{\mathfrak{M}}$ be the space of all $\widetilde{\phi}$ defined for $(x,\xi) \in \mathbb{R}^2$ which are 1-periodic in x, satisfy

$$\widetilde{\phi}(x,\xi+n) = e^{-2\pi inx}\widetilde{\phi}(x,\xi) \text{ for } n \in \mathbb{Z}, \tag{4}$$

and satisfy $\|\widetilde{\phi}\|_{\widetilde{\mathfrak{M}}} = \left(\int_0^1\int_0^1 |\widetilde{\phi}(x,\xi)|^2 dx d\xi\right)^{1/2} < \infty$. Similarly, it is easy to show that $\widetilde{Z}$ is an isometry onto $\widetilde{\mathfrak{M}}$.

Observe that the mapping $U : \mathfrak{M} \to \widetilde{\mathfrak{M}}$ defined by $(U\phi)(x,\xi) = e^{-2\pi ix\xi}\phi(x,\xi) := \widetilde{\phi}(x,\xi)$ is a unitary isometry onto $\widetilde{\mathfrak{M}}$. It is also clear that $Z^{-1} : \mathfrak{M} \to L^2(\mathbb{R})$ and $\widetilde{Z}^{-1} : \widetilde{\mathfrak{M}} \to L^2(\mathbb{R})$ are given by $\int_0^1 \phi(x,\xi)d\xi$ and $\int_0^1 \widetilde{\phi}(x,\xi)dx$, respectively.

Let us write out explicitly the "formulae" of what we have done:

$$(UZf)(x,\xi) = e^{-2\pi ix\xi}\sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi ik\xi} = \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i(x+k)\xi}. \tag{5}$$

It follows that

$$(\widetilde{Z}^{-1}UZf)(\xi) = \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi i(x+k)\xi}dx = \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx = \widehat{f}(\xi).$$

That is, we have the following factorization for the Fourier transform, $\mathfrak{F}$, (and by similar arguments $\mathfrak{F}^{-1}$),

$$\mathfrak{F} = \widetilde{Z}^{-1}UZ \text{ and } \mathfrak{F}^{-1} = Z^{-1}U^*\widetilde{Z}. \tag{6}$$

This gives us explicit formulae for the Fourier transform and its inverse on $L^2(\mathbb{R})$ in terms of these simple operators.

Believe us that it follows that many important Plancherel-related results are obtained in surprisingly simple ways by using these Zak transforms.[3] Let us cite, for example, the following expression for the bracket:

$$[\phi, \psi](\xi) = \int_0^1 (Z\phi)(x, \xi)\overline{(Z\psi)(x, \xi)}dx, \tag{7}$$

when $\phi, \psi \in L^2(\mathbb{R})$. This shows that the bracket has a basic connection with the Zak transforms.

Let us present two more results of how a property of $\mathcal{B}_\phi$ is equivalent to a property of the weight p_ϕ (they are much more challenging to prove than the aforementioned properties):

(iv) *Schauder bases.* A well known result of Hunt, Muckenhoupt, and Wheeden,[4] describes $A_q(\mathbb{T})$ *weights*, spaces which are important in singular integral theory. In particular, p is an A_2 weight if and only if

$$\left(\frac{1}{|I|}\int_I p(\xi)d\xi\right)\left(\frac{1}{|I|}\int_I \frac{1}{p(\xi)}d\xi\right) \leq M < \infty \tag{8}$$

for any interval $I \subset [0, 1)$, where M is some constant independent of I. Nielsen and Šikić,[5] showed that $\mathcal{B}_\phi$ is a *Schauder Basis* for $\langle\phi\rangle$ if and only if p_ϕ is an A_2 weight. Recall that a Schauder basis in the Hilbert space $\langle\phi\rangle$ is one such that, for each $f \in \langle\phi\rangle$, there exists a $\mathbb{C}$-valued sequence $(a_k)_{k\in\mathbb{Z}}$ such that

$$f = \lim_{K\to\infty} \sum_{|k|\leq K} a_k(f)\phi(\cdot - k),$$

the limit being in the $\langle\phi\rangle$ norm (we also make the requirement that each $a_k : \langle\phi\rangle \to \mathbb{C}$ be a bounded linear functional for all $k \in \mathbb{Z}$).

(v) $\ell^2(\mathbb{Z})$ *independence.* Kolmogoroff introduced the notion of "$\ell^2(\mathbb{Z})$ linear independence" for a sequence of vectors in, say, a separable Hilbert space $\mathcal{H}$. Let $(\phi_k)_{k\in\mathbb{Z}}$ be the sequence; if, for a sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\lim_{N\to\infty} \sum_{|k|\leq N} a_k\phi_k = 0,$$

with convergence in the $\mathcal{H}$ norm, then $a_k = 0$ for all $k \in \mathbb{Z}$ — this is what it means for the sequence to be $\ell^2(\mathbb{Z})$ independent. Then Saliani[6] and Paluszyński[7] showed that it is true that $\mathcal{B}_\phi$ (with $\phi_k = \phi(\cdot - k)$ for a nonzero ϕ) is $\ell^2(\mathbb{Z})$ independent if and only if $p_\phi(\xi) > 0$ almost everywhere.

An important point we want to make is that the principal shift invariant spaces play an important role in several areas of Analysis. Let us consider the well known result of Shannon[8] in Sampling Theory:

Let $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$ for $x \in \mathbb{R}$. It is easy to see that $\widehat{\text{sinc}}(\xi) = \chi_{[-1/2,1/2)}(\xi)$. If we let $\langle\phi\rangle = \langle\text{sinc}\rangle$, then the result of Shannon asserts that $f \in \langle\text{sinc}\rangle$ if and only if

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\text{sinc}(x - k), \tag{9}$$

where the convergence of the series is uniform, absolute, and in $L^p(\mathbb{R})$, $p \geq 2$ (this follows from the band limited property of sinc — since $\widehat{\text{sinc}} = \chi_{[-1/2,1/2)}$). It is clear from Equation 9 that all values of f are known if we know the values of f on $\mathbb{Z}$ (the countable "sampling" set). It is easy to see that such a result could be quite important in applications. One can see the truth of Equation 9 if we observe that ϕ satisfies the Nyquist condition:

$$\phi(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} . \tag{10}$$

It is natural to ask if there is a more general condition for the sequence $(\phi(k))_{k\in\mathbb{Z}}$ that produces a "sampling" result (that all values of $f \in \langle\phi\rangle$ are determined by the values on $\mathbb{Z}$). Using our knowledge of the principal shift invariant spaces and the Zak transforms we discovered such a result as well as others.[9] One can replace the Nyquist condition on the sequence $c = (c_k)_{k\in\mathbb{Z}} = (\phi(k))_{k\in\mathbb{Z}}$ by using the convolution of sequences in $\ell^2(\mathbb{Z})$: $(c * d)_k = \sum_{j\in\mathbb{Z}} c_{k-j} d_j$. This operation is well defined on this space. In particular, there are infinitely many sequences (c_k) satisfying the *convolution idempotent property*:

$$c * c = c. \tag{11}$$

For example, consider the sequence of Fourier coefficients of χ_E where E is any measurable subset of $[0, 1)$ (and extend 1-periodically). Each such sequence satisfies Equation 11 since the square of a characteristic function is simply itself. One can show[9] that the Nyquist condition can be replaced by

$$\phi(k) = c_k \text{ for } (c_k) \text{ satisfying Equation 11.} \tag{12}$$

Moreover, if ψ is an $L^2(\mathbb{R})$ function then, under suitable restrictions,[9] the space $\langle\psi\rangle$ contains a sampling function ϕ with $\langle\psi\rangle = \langle\phi\rangle$; the function ϕ comes explicitly in terms of the Zak transform and ψ.

The "ubiquitous" nature of the principal shift invariant spaces is encountered in many other areas, and their properties are most useful for the study of these areas. Consider *Wavelets*. In one dimension, a (classical) *wavelet* ψ is an element of $L^2(\mathbb{R})$ such that the set $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ where $\psi_{jk} = 2^{j/2}\phi(2^j x - k)$, is an orthonormal basis of $L^2(\mathbb{R})$. That is, we first form the system $\mathcal{B}_\psi$ and, then, apply the dilation $(D_j f)(x) := 2^{j/2} f(2^j x)$ for $j \in \mathbb{Z}$ to each member $f \in \mathcal{B}_\psi$. An important method for constructing such wavelets is known as the MRA method (with MRA standing for multiresolution analysis).

An MRA consists of a sequence of closed subspaces V_j, $j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ satisfying the following:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.

(b) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$.

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(d) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(e) There exists a function $\phi \in V_0$ so that $\mathcal{B}_\phi = \{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0. This ϕ is called the *scaling function* of this MRA.

We thus meet the principal shift invariant space $V_0 = \langle\phi\rangle$. A very good example of an MRA and how it produces a wavelet is given to us by the Shannon sampling theorem. Let us see how it provides us with a wavelet produced by an MRA. We call this wavelet the Shannon wavelet. Let ϕ be the sinc function and define $V_0 = \langle\phi\rangle$. Define the V_j to be the dilations of the V_0, i.e. $V_j = D_j(V_0)$. That this sequence of subspaces is an MRA is quite immediate since $\widehat{V_0} = L^2([-1/2, 1/2))$.

Consider the set $E = [-1, -1/2) \cup [1/2, 1)$, and define ψ by $\widehat{\psi} = \chi_E$. Clearly the various dyadic dilations of E, namely the sets $2^j E$, produce a disjoint covering of $\mathbb{R}\backslash\{0\}$; also, the sets

$$\widehat{W_j} = \overline{\operatorname{span}\{\chi_E(2^{-j}\xi)e^{2\pi i k 2^{-j}\xi} : k \in \mathbb{Z}\}} = L^2(2^j E) \tag{13}$$

are mutually orthogonal with $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$. Moreover,

$$V_{j+1} = V_j \oplus W_j \tag{14}$$

for all $j \in \mathbb{Z}$ (in other words, the orthogonal complement of V_j within V_{j+1} is W_j). It is completely straightforward from these observations that ψ is actually a wavelet. This Shannon wavelet is probably the simplest example of a wavelet, as the previous simple argument demonstrates, even though the Haar wavelet is traditionally cited as the simplest.

Let us give a quick description of the MRA construction of wavelets.[10] Let ϕ be the scaling function of a general MRA. Since the Fourier transform of the dilation operator D_j is D_{-j} — in the sense that $\widehat{D_j \phi} = D_{-j}\widehat{\phi}$ — we have that $\widehat{\phi}(2\xi)$ belongs to $V_{-1} \subset V_0$. Let $\widehat{W_{-1}}$ be the orthogonal complement of $\widehat{V_{-1}}$ in $\widehat{V_0}$. Clearly there exists a 1-periodic function, h_0, in $L^2(\mathbb{T})$ such that $\widehat{\phi}(2\xi) = h_0(\xi)\widehat{\phi}(\xi)$. The function h_0 is called the *low-pass filter*, and one constructs another function, $h_1(\xi) = e^{2\pi i\xi}\overline{h_0(\xi + \frac{1}{2})}$, called the *high-pass filter*. This pair of functions satisfies

$$|h_0(\xi)|^2 + |h_1(\xi)|^2 = 1. \tag{15}$$

Furthermore, $\widehat{\psi}(2\xi) = h_1(\xi)\widehat{\phi}(\xi)$ gives us the desired wavelet: $\psi \in W_0$, with $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_0. This means that the principal shift invariant space W_0 has the property that its dilations $D_j W_0 = W_j$ form an orthogonal decomposition of $L^2(\mathbb{R})$ (see Equation 13). We see, therefore, that the Shannon wavelet is a simple example of an MRA wavelet obtained from the scaling function sinc. Observe that V_0 and W_0, in general, have the following different properties: the dilations V_j of V_0 form an increasing sequence of subspaces of $L^2(\mathbb{R})$ while $\{W_j : j \in \mathbb{Z}\}$ is a mutually orthogonal sequence of subspaces.

The last condition in the definition of MRA, namely that $\mathcal{B}_\phi$ be an orthonormal basis for V_0 is quite restrictive from a pragmatic point of view, whether one is interested in pure or applied Analysis. Even when one asks whether $\mathcal{B}_\psi$ is a frame or Riesz basis (defined below) for $\langle \psi \rangle$, the weight function p_ψ still gives an extremely simple answer:

(vi) *Frames.* Recall that a frame for a Hilbert space H is a countable subset $\{f_k\}$ so that there are finite, positive constants A and B so that for every $f \in H$

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2.$$

It can be shown[2] that $\mathcal{B}_\psi$ is a frame with constants A and B if and only if $A\chi_\psi \leq p_\psi \leq B\chi_\psi$ almost everywhere, where χ_ψ is the characteristic function of the support of p_ψ. Thus, in particular, $\mathcal{B}_\psi$ is a Parseval frame (a frame with $A = B = 1$) if and only if p_ψ is the characteristic function of a subset of $[0, 1]$.

(vii) *Riesz bases.* Recall that a Riesz basis is the image of an orthonormal basis under a bounded, invertible linear transformation. It follows that $\mathcal{B}_\psi$ is a Riesz basis if and only if there are finite, positive constants A and B so that $A \leq p_\psi \leq B$ almost everywhere.[2]

1.1 Non-principal shift invariant spaces in $\mathbb{R}$

In all of the above discussions, our dilations were dyadic. One may well wonder whether it makes a difference if one replaces 2^j with 3^j or, more generally, $(p/q)^j$ for positive integers $p > q$ which are relatively prime. In fact, such a change makes the corresponding notion of an MRA more complex. The primary complication is that property (e) from the above definition of MRA is no longer achievable. Instead, one must broaden one's view to considering the case $V_0 = \langle \phi_1, \phi_2, ..., \phi_{p-q}\rangle$, where this latter symbol means "the smallest shift-invariant space containing all the functions $\phi_1, ..., \phi_{p-q}$". In such a case, one does not produce a single wavelet, but rather a vector of $p - q$ functions called a multiwavelet. In the ensuing analysis, rather than a single low-pass filter, one produces a matrix of low-pass filters $M_0(\xi)$, so that

$$\begin{pmatrix} \phi_1(\frac{p}{q}\xi) \\ \phi_2(\frac{p}{q}\xi) \\ \cdots \\ \phi_{p-q}(\frac{p}{q}\xi) \end{pmatrix} = M_0(\xi) \begin{pmatrix} \phi_1(\xi) \\ \phi_2(\xi) \\ \cdots \\ \phi_{p-q}(\xi) \end{pmatrix}.$$

Thus one is rather naturally drawn to consider "wavelets" (read: multiwavelets) generated by more complicated shift-invariant spaces. If one considers the dyadic definition of MRA while relaxing condition (e) to allow for V_0 to be generated by more than one function, there is, for example, a family of multiwavelets described by Journé (see Example A on page 350 of the book by Hernández and Weiss[10]) requiring V_0 to be a shift-invariant space generated by n functions. In fact, D. Bakić has produced a more complicated example where V_0 must actually be infinitely generated.

2. IN SETTINGS BEYOND $\mathbb{R}$

2.1 The $\mathbb{R}^n$ case

To describe some of the complications one encounters in shift invariant spaces in $\mathbb{R}^n$ — in particular, to generalize the notion of an MRA wavelet to $\mathbb{R}^n$ — first observe that while one could simply view $L^2(\mathbb{R}^n)$ as, roughly speaking, the tensor product of n copies of $L^2(\mathbb{R})$ and import the one-dimensional theory into n-dimensions, such an approach dramatically increases the numerical complexity for applied problems. Thus one would like a "naturally" n-dimensional framework.

With that in mind, one would obviously like to use $\mathbb{Z}^n$ as the family of translations, but it is not immediately obvious how to "best" replace the dilations, even in $\mathbb{R}^2$. It is fairly common to use the quincunx matrix,

$$q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In this setting, with q producing dilations and $\mathbb{Z}^n$ as the translations, suppose that we were to attempt to generalize the Haar wavelet from $L^2(\mathbb{R})$: this is just the standard (dyadic, one-dimensional) MRA wavelet with scaling function $\phi = \chi_{[0,1]}$. It is possible[11] to generalize the Haar wavelet to $\mathbb{R}^2$ — that is, a $(q, \mathbb{Z}^2)$-MRA wavelet whose properties are very similar to those of the Haar wavelet. However, the scaling function produced in such a case is the characteristic function of a fractal set called "the twin dragons", and so mostly useful only in the abstract.

By including not just dilations by q, but a group of dilations generated by q and the symmetry group of the square (a dihedral group), one can produce a Haar-like wavelet whose scaling function is the characteristic function of a right triangle.[12] Alternatively, one could use just q to generate the dilations with the caveat that V_0 is generated by several characteristic functions (one coming from each element of the square symmetry group acting on the aforementioned triangle). Thus one again encounters the more complicated shift-invariant spaces described at the end of the previous section.

2.2 Locally compact abelian groups

In this section, we summarize the results of recent work on locally compact abelian groups.[13] The general idea is that one can appropriately generalize the notions of translation, principal shift invariant spaces, brackets, and weight functions from the $\mathbb{R}$ setting to the locally compact abelian setting in such a way that one obtains a similar dichotomy between principal shift invariant spaces and their weight functions.

Suppose that G is a locally compact abelian (LCA) group. Basic facts about harmonic analysis on such groups can be found in texts by Rudin[14] or Folland,[15] including the definition and various properties of the Fourier transform in this setting. We shall use additive notation for the group G. Recall that a *character* of G is a continuous map $\alpha : G \to \mathbb{C}$ such that

$$|\alpha(g)| = 1 \text{ for all } g \in G, \text{ and } \alpha(g_1 + g_2) = \alpha(g_1) \cdot \alpha(g_2) \text{ for all } g_1, g_2 \in G.$$

The *character group* of G is the multiplicative group of all characters of G. We take the *dual group* $\widehat{G}$ to be the character group of G — one has the famous examples

$$\widehat{T^n} = \{e^{2\pi i \xi \cdot k} : k \in \mathbb{Z}^n\} \text{ and } \widehat{Z^n} = \{e^{2\pi i \xi \cdot k} : k \in \mathbb{T}^n\}.$$

A *representation* of an LCA group G on a Hilbert space H is a strongly continuous map $g \to T_g$ from G into $\mathcal{L}(H, H)$, the group of bounded linear operators on H having bounded inverses, such that $T_g \circ T_h = T_{g+h}$ for all $g, h \in G$. We say that a representation is *unitary* if all the operators T_g are unitary on H.

One key example to keep in mind is the *regular representation* $h \to R_h$ on $L^2(G, dg)$ given by $(R_h\phi)(x) = \phi(x + h)$.

First, we generalize the notion of translation, shift invariance, and principal shift invariant subspaces. If T is a unitary representation of G over H, a closed linear subspace S of the Hilbert space H is said to be T-invariant if $T_g(S) \subset S$ for all $g \in G$. Given $\psi \in H \backslash \{0\}$, define the closed linear subspace of H generated by ψ to be

$$\langle \psi \rangle = \overline{\operatorname{span}\{T_g \psi : g \in G\}}^H. \tag{16}$$

The subspace $\langle \psi \rangle$ is clearly T-invariant and is called the *cyclic* T-invariant subspace generated by ψ. For the regular representation, T_g represents translation by g, T-invariance is just translation invariance, and cyclic subspaces are just the obvious generalization of principal shift invariant spaces to this setting.

Now we generalize the bracket. Fix a Haar measure on $\widehat{G}$. A unitary representation T of an LCA group G on a Hilbert space H is said to be *dual integrable* if there exists a function, which we shall call *bracket*,

$$[\cdot,\cdot] : H \times H \to L^1(\widehat{G}, d\alpha)$$

such that

$$\langle \phi, T_g \psi \rangle_H = \int_{\widehat{G}} [\phi, \psi](\alpha) \overline{\alpha(g)} d\alpha.$$

The notion of dual integrability, i.e. that the integration takes place over the dual group $\widehat{G}$, is the key idea here. For the regular representation, using the Fourier transform on G and its Plancherel identity, one has

$$\langle \phi, R_h \psi \rangle_{L^2(G,dg)} = \int \phi(g) \overline{(R_h \psi)(g)} dg = \int_{\widehat{G}} \widehat{\phi}(\alpha) \overline{\widehat{R_h \psi}(\alpha)} d\alpha = \int_{\widehat{G}} \widehat{\phi}(\alpha) \overline{\widehat{\psi}(\alpha)}\, \overline{\alpha(h)} d\alpha. \tag{17}$$

Thus for the regular representation, the bracket is simply given by $[\phi, \psi](\alpha) = \widehat{\phi}(\alpha) \overline{\widehat{\psi}(\alpha)}$, which is reminiscent of the bracket on $\mathbb{R}$. This notion of bracket is as useful as the 1-dimensional version. In particular, we have the property that $\langle \phi \rangle \perp \langle \psi \rangle$ if and only if $[\phi, \psi] = 0$ (almost everywhere). Moreover, one has the following general theorems:

THEOREM 2.1. *Let $g \to T_g$ be a dual integrable unitary representation of an LCA group G on a Hilbert space H. For $\psi \in H \backslash \{0\}$, define $\Omega_\psi := \{\alpha \in \widehat{G} : [\psi, \psi](\alpha) > 0\}$. Then the map*

$$J_\psi(\phi) = 1_{\Omega_\psi} \frac{[\phi, \psi]}{[\psi, \psi]} \text{ for } \phi \in H$$

is a linear, one-to-one isometry from $\langle \psi \rangle$ onto the weighted space $L^2(\widehat{G}, [\psi, \psi](\alpha) d\alpha)$.

Thus, as in the one-dimensional case, the space $\langle \psi \rangle$ is isometrically isomorphic to a weighted L^2 space with the weight coming from the bracket. In Section 4 of the article referenced at the beginning of this section, explicit computations are performed using the integer translations as the representation to produce the isomorphism from the one-dimensional setting, i.e. the isomorphism given in Equation (2) above. Other computations using the Gabor representation shows the connection with the Zak transform given in Equation (7) above.

THEOREM 2.2. *Let G be a* countable *abelian group and let $k \to T_k$ be a dual integrable unitary representation of G on a Hilbert space H. Let $\psi \in H \backslash \{0\}$. Let $\mathcal{B}_{\psi,G} := \{T_k \psi : k \in G\}$ and $\langle \psi \rangle$ as above. Then one has the following:*

(i) $\mathcal{B}_{\psi,G}$ is an orthonormal basis for $\langle \psi \rangle$ if and only if $[\psi, \psi](\alpha) = 1$ for almost every $\alpha \in \widehat{G}$.

(ii) $\mathcal{B}_{\psi,G}$ is a Riesz basis for $\langle \psi \rangle$ with constants A and B if and only if $A \leq [\psi, \psi](\alpha) \leq B$ for almost every $\alpha \in \widehat{G}$.

(iii) $\mathcal{B}_{\psi,G}$ is a frame for $\langle \psi \rangle$ with constants A and B if and only if $A \leq [\psi, \psi](\alpha) \leq B$ for almost every $\alpha \in \Omega_\psi := \{[\psi, \psi](\alpha) > 0\}$. In particular, $\mathcal{B}_{\psi,G}$ is a Parseval frame for $\langle \psi \rangle$ if and only if $[\psi, \psi] = \chi_{\Omega_\psi}$.

2.3 Non-abelian groups

Recent work by Barbieri, Hernández, and Mayeli[16] generalizes shift-invariant spaces and the bracket to the non-abelian Heisenberg groups $\mathbb{H}_n$ (and actually to a broader class of "similar" non-abelian groups). The Heisenberg groups are $2n+1$-dimensional groups parametrized by $(x_1, ..., x_n, y_1, ..., y_n, t)$, where the group law is given by

$$(x_1, ..., x_n, y_1, ..., y_n, t) \cdot (x'_1, ..., x'_n, y'_1, ..., y'_n, t') = (x_1 + y_1, ..., x_n + y_n, t + t' + x_1 y'_1 + ... + x_n y'_n);$$

the non-commutativity comes from the last term. The group law is better understood by considering the upper-triangular matrices

$$\begin{pmatrix} 1 & \mathbf{x}^T & t \\ 0 & I_n & \mathbf{y} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}.$$

The concept of shift-invariant space comes by choosing, for example, the translations L_γ as coming from left translation by the family Γ of matrices above when all the entries are integers — their paper works equally well for other similar lattices, but this is the motivating example.

Harmonic analysis in this setting is more complicated than the abelian case; in particular, the character group is insufficient to support a Fourier transform, and one must exert effort to parametrize the (correct) dual space and compute an appropriate measure to support a Plancherel identity. However, once one is comfortable in this abstract setting, one is able to produce a bracket function $[\cdot, \cdot] : L^2(\mathbb{H}_n) \times L^2(\mathbb{H}_n) \to L^2([0, 1))$:

$$[\phi, \psi](\alpha) = \sum_{j \in \mathbb{Z}} \langle \mathfrak{F}_{\mathbb{H}_n} \phi(\alpha + j), \mathfrak{F}_{\mathbb{H}_n} \psi(\alpha + j) \rangle_{\mathcal{HS}} \, |\alpha + j|^n,$$

where $\mathfrak{F}_{\mathbb{H}_n}$ is the Fourier transform on the Heisenberg group (which takes values in a space of Hilbert-Schmidt operators), the inner product is the Hilbert-Schmidt inner product, and the last factor is a rescaling factor coming from the "Plancherel measure" in this setting.

While significantly more difficult to interpret, this bracket enjoys many of the same properties of its relatives on locally compact abelian groups. For example, Barbieri, Hernández, and Mayeli were able to prove the following theorems:

Theorem 2.3. *The collection $\{T_\gamma \psi \ \gamma \in \Gamma\}$ is a frame for $\langle \psi \rangle_{\Gamma, L}$ with constants A and B if and only if $A \leq [\psi, \psi](\alpha) \leq B$ for almost every α in the support of $[\psi, \psi]$.*

Theorem 2.4. *Let Ω_ψ denote the support of $[\psi, \psi]$. The map*

$$(S_\psi(\phi))(\alpha) := \chi_{\Omega_\psi}(\alpha) \frac{[\phi, \psi](\alpha)}{[\psi, \psi](\alpha)}$$

is an isometry from an appropriate principal shift-invariant space onto the weighted space $L^2([0, 1), [\psi, \psi](\alpha) d\alpha)$.

In the previous theorem, the shift invariant space is not generated by the full family of translations Γ described above, but we refer the reader to the paper in question for the full details.

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